MATH 6326: PARTIAL DIFFERENTIAL EQUATIONS

Year Course offered: Fall 2021

Department: Mathematics

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Prerequisites:

- Graduate standing, Introduction to Real Analysis (Math 4331) or equivalent.
- Recommended, but not required: Math 6320-21

Course Description: This course introduces four main types of partial differential equations: parabolic, elliptic, hyperbolic and transport equations. The focus is on existence and uniqueness theory. Maximum principles and regularity of solutions will be considered. Other concepts that will be explored include weak formulations and weak solutions, distribution theory, fundamental solutions. The course will touch on applications and a brief introduction to numerical methods: finite differences, finite volume, and finite elements.

References:

- Robert McOwen, *Partial Differential Equations, Methods and Applications*, 2nd Ed. (2004)
- Lawrence C. Evans, *Partial Differential Equations*, Graduate studies in mathematics 19.2 (1998).

0.1. **Topics.**

I. Introduction
   a) Classification of PDE
   b) Examples of PDE in physics:

      Elastic deformation of a string, Elastic deformation of a membrane, String vibration, Heat Diffusion, Compressible fluid flow (Traffic flow problem).

II. Parabolic Equations/ Heat equation in bounded domains

a) Existence and Uniqueness:
   (i) A basis for $L^2[0,1]$:
   (ii) Uniqueness of solutions via eigenfunction expansion.

   Highlight connection to spectral methods, which are generally used to compute solutions to reaction-diffusion equations.

b) Stability via Parseval's and Monotone Convergence Theorem
c) Regularity of solutions
d) Maximum principles

III. Finite Difference approximations
- Explicit schemes Consistency, Order, Convergence (conditional)
- Theta schemes and define/show consistency, order, stability, convergence of the scheme.

IV. Heat equation on $\mathbb{R}^n$
a) Weak derivatives, Distribution theory, convolutions, fundamental solutions
b) Fourier Transform
c) Heat Kernel
d) Regularity
e) Applications: Brownian motion and viscous incompressible flow

V. Hyperbolic equations/ Transport Equations
a) Introduction, definition and examples, including wave equation.
b) Transport equations as preliminary case, method of characteristics
c) Scalar conservation laws and jump conditions. Method of characteristics fails when u is not smooth.
d) Weak formulation for quasilinear equations, allows for solutions with jumps
e) Rankine-Hugoniot condition
f) Weak solutions are not unique, example Burger's equation.
g) Viscosity solutions for burgers equations to show existence

VI. Finite Volume methods:
Generalities, Lax-Friedrichs, some numerical results

VII. Elliptic equations
a) Introduction and examples, model problem
b) Weak formulation, Lax-Milgram, Riesz Representation theorem
c) Distributions, Sobolev spaces, Poincaré inequality, traces, H^1_0, Greens functions
d) Rellich compactness theorem
e) Homogeneous Dirichlet problem: $-\Delta u + u = f$
f) Regularity: If $f \in H^m(\Omega)$ and $\partial\Omega$ is of class C^{m+2}. Then the solution to $-\Delta u + u = f$ with homogenous Dirichlet BC is in $u \in H^{m+2}(\Omega)$

g) Some Sobolev embeddings: $H^m(\Omega) \subset C^k(\overline{\Omega})$.
h) Maximum principles
i) Non homogenous Dirichlet: $-\Delta u + u = f$
j) Neumann Homogeneous: $-\Delta u + u = f$

VIII. Finite Elements: some numerical results